About Minimal Surfaces in Three-Dimensional Galilean Space.

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Abstract— In this article, minimal surfaces in Galilean space are studied. The circumference of the parabolic point of the surface has been studied, and it has been proven that a special parabolic surface is minimal. It has been shown that the cyclic surface is minimal and its equation has been derived. The conditions for the existence of a minimum surface are defined and the equations are shown when the surface is given by an explicit and vector equation.

Keywords—Galilean geometry, minimum surface, cyclic surface, special parabolic point, mean curvature.

I. Introduction

The geometry of Galilean space belongs to the geometry of spaces with degenerate metrics. A. Artikbaev [1] and N.M. Makarova [2] were first engaged in solving specific problems of geometry "in general" in Galilean space.

In the study of the geometry of non-Euclidean spaces we sometimes use the method of superimposed space, that is, the coordinate system of non-Euclidean space is considered as the coordinate system of Euclidean space [3;4]. If the coordinate system of Galilean space is considered as a Euclidean system, then some of our results about isometry are a generalization of the notion of "isometry of surfaces by section" studied in the works of A. Sharipov [5,6].

After 2000, the geometry of Galilean space began to be widely studied [7;8;9]. In these works, the differential geometry of Galilean space was studied. The degeneracy of the Galilean space metric does not make it possible to study an arbitrary surface in Galilean space. Therefore, we consider only surfaces that do not have special reference planes.

In Galilean space there is a "cyclic surface" introduced by A. Artikbayev. Artikbayev "cyclic surface". The main properties of the "cyclic surface" are studied by E. Kurbanov [10]. In this article, we'll also divide the parabolic points of the surface into two.

The saddle-shaped surface in the Galilean space retains its uniqueness in the Euclidean space. A cyclic surface in Galilean space is a saddle-shaped surface in Euclidean space. In the study of the geometry of the cyclic point of the surface, it is necessary to consider some new issues first, the geometry of the saddle surfaces in the Euclidean space and the geometry of the isotropic cone in the pseudo-Euclidean space are applied to the study of the cyclic surface in the Galilean space.

It is known that in Euclidean space minimal surfaces are defined as surfaces with zero mean curvature. These surfaces have the smallest area among surfaces with common edges that are a given closed contour.

In this article, we will also divide the parabolic points of the surface into two parts and show the conditions for the minimum length of the surface, the minimality of a surface whose points are cyclic and specifically parabolic.

II. Preliminaries

Let a three-dimensional affine space A_3 be given, $Oxyz$ be a system of affine coordinates with the origin at point $O(0,0,0)$ and $\{\vec{\imath},\vec{\jmath},\vec{k}\}$ be the basis vectors in this space.

The scalar product of vectors $\vec{X}\{x_1,y_1,z_1\}$ and $\vec{Y}\{x_2,y_2,z_2\}$ is determined by formula

$$(\vec{X}\vec{Y})=\begin{cases}x_1x_2, if\ \ x_1x_2\neq 0,\\ y_1y_2+z_1z_2, if\ x_1x_2=0.\end{cases} \quad (1)$$

Definition 1. An affine space in which the scalar product of vectors $\vec{X},\vec{Y}$ is defined by formula (1) is called a Galilean space and is denoted by G_3 [11;12].

Let us find out the geometric meaning of the distance between two points of Galilean space G_3, defined as the norm of the vector connecting the points between which the distance is determined.

Let points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ be points of Galilean space G_3, and $x_1\neq x_2$. Then vector $\overrightarrow{AB}$

$$\overrightarrow{AB}\{x_2-x_1,y_2-y_1,z_2-z_1\}$$

and

$$AB=\left|\overrightarrow{AB}\right|=\sqrt{(\overrightarrow{AB}\cdot\overrightarrow{AB})}=|x_2-x_1|.$$

The distance between points A and B is equal to the length of the projection of vector $\overrightarrow{AB}$ onto axis 0x (see Fig. 1).

If $x_1=x_2=x_0$, then vector $\overrightarrow{AB}$ is parallel to plane $0yz$, and the distance between points $A(x_0,y_1,z_1)$ and $B(x_0,y_2,z_2)$ is determined by formula

$$AB=|\overrightarrow{AB}|=\sqrt{(y_2-y_1)^2+(z_2-z_1)^2}.$$

Obviously, points A and B lie on the plane $x=x_0$, and the distance will be the Euclidean distance between the corresponding points.

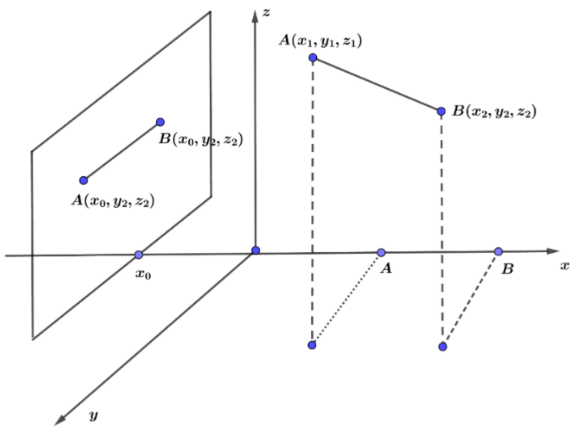

Figure 1.

Therefore, the geometry on the plane $x = \mathrm{x}_0$ of Galilean space will be Euclidean; such planes are called special planes of Galilean space [11].

Let F be the surface of space G_3 without special tangent planes. Let us introduce a special system of curvilinear coordinates. To do this, consider all possible intersections F with special planes $x = const$.

Let us choose as curvilinear coordinates $u = u_0$ a family of curves formed by intersections of the surface with special planes, and as coordinate lines $v = v_0$ – arbitrary lines forming a network on the surface F. With this choice of curvilinear coordinates, the surface equations have the form

$$\vec{r} = \vec{r}(u, v) = u\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}.$$

In this case, vectors $\vec{r}_u, \vec{r}_v$ form a basis in the tangent plane of the surface, which is Galilean. The direction of vector $\vec{r}_v$ corresponds to the selected direction of the Galilean plane.

The vector equation of a line in Galilean space will look like this:

$$\vec{r} = \vec{r}(s) = s\vec{i} + y(s)\vec{j} + z(s)\vec{k}.$$

Let a curve with equation $v = v(u)$ be given on surface F. Consider the length of the curve on the surface. Calculating the arc length of a curve segment with ends at points $A(u_0)$ and $B(u_1)$, where $u_1 \neq u_0$, we obtain that the differential of the arc length is $ds = |\vec{r}_u du + \vec{r}_v dv|$. Consequently, the square of the differential of the curve arc on the surface is equal to the square of the increment of coordinate

$$ds^2 = du^2.$$

We call the resulting from the first fundamental form of the surface [8]. When $du = 0$, we have $u = const$. In this case, the curve lies on a special plane.

The differential of the arc length of a curve is calculated using formula

$$ds_2^2 = (y_v^2 + z_v^2)dv^2 = G(u, v)dv^2,$$

where ds_2^2 is the first additional fundamental form of the surface.

Consequently, for the chosen curvilinear coordinate, the coefficients of the first fundamental form have the form

$$E(u, v) = 1, G(u, v) = y_v^2 + z_v^2. \quad (2)$$

Mean surface curvature

$$2H = N \quad (3)$$

and total surface curvature

$$K = \frac{LN - M^2}{G(u, v)},$$

where

$$L = (\vec{r}_{uu}\vec{n}) = \frac{y_{uu}z_v - z_{uu}y_v}{\sqrt{G(u,v)}},$$
$$M = (\vec{r}_{uv}\vec{n}) = \frac{y_{uv}z_v - z_{uv}y_v}{\sqrt{G(u,v)}}, \quad (4)$$
$$N = (\vec{r}_{vv}\vec{n}) = \frac{y_{vv}z_v - z_{vv}y_v}{\sqrt{G(u,v)}}.$$

are the coefficients of the second fundamental form [11]. The principal curvatures of the surface are defined as follows:

$$k_1 = a_{11} = L - \frac{M_2}{N}, \ k_2 = a_{22} = N.$$

When Artykbaev classified the points of surfaces, he proved that if the conditions $N = 0$, $M \neq 0$ for the coefficients of the second fundamental form are fulfilled at all points of the surface, then this surface is a cyclic surface [13]. At $N = 0$ the mean curvature of $H = 0$ and $K = -M^2 < 0$ the Gaussian curvature is negative.

Let a regular surface be given in Galilean space. We take a point M on this surface and intersect the surface through this point with a special plane. When we cut, we get a curve $u = const$. If the tangent vector of the point M of this curve coincides with the asymptotic direction, then this point is called a cyclic point.

Definition-2. A surface whose points are all cyclic is called a cyclic surface.

We define the minimal surfaces of Galilean space as well as in Euclidean space as a surface of mean curvature which converges to zero. Then we have $N = 0$.

It is easy to prove that the surface with zero mean curvature in G_3 has the property of a minimal surface in G_3. The area of the minimal surface will be the smallest among the surfaces with a common edge in G_3.

Indeed, suppose that D is a convex region in the plane of general position (i.e., $z = 0$) and γ is the boundary of this region. Consider a spatial curve $\bar{\gamma}$ mutually uniquely projecting onto γ. The area of the surface F uniquely projecting onto the region D with edge $\bar{\gamma}$ is calculated by formula

$$S = \iint_D \sqrt{G(u, v)}\, dudv,$$

where $G(u, v)$ is the coefficient of the first quadratic form of the surface F. Since the area D is convex, the double integral can be calculated using repeated integrals, i.e.

$$S = \iint\limits_D \sqrt{G(u, v)}\, dudv = \int_a^b \left(\int_{\varphi_1(\alpha)}^{\varphi_2(\alpha)} \sqrt{G(u, v)} dv \right) du.$$

Expression $\int_{\varphi_1(\alpha)}^{\varphi_2(\alpha)} \sqrt{G(u, v)} dv$ - in the plane $x = u = const$ gives the arc length of the curve γ, formed by the intersection of the special plane with the given surface.

Hence, the surface area is directly proportional to the arc length of the curve; since the edge of the surface $\bar{\gamma}$ has only two points in common with the plane $x = const$, the length of the curve $\bar{\gamma}$ connecting these points will be smallest when $\bar{\gamma}$ is the segment connecting these points. Furthermore, the coefficient N is proportional to the curvature of the curve γ. Equality to zero of N means that γ is a segment. So, minimality of area is achieved only in this case. Hence the following statement can be made.

III. My results

In [13] the point of the surface where the condition $k_1 = a_{11} = 0$, $k_2 = a_{22} \neq 0$ is fulfilled is called parabolic. In this case, the surface indicatrix in Galilean space has the form

$$Ny'^2 = \pm 1$$

or is reduced to this form (see Fig. 2).

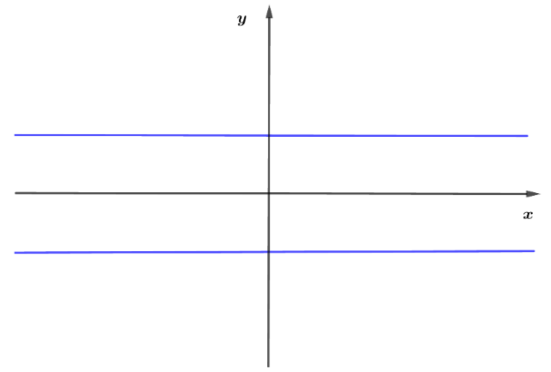

Figure 2.

In addition, we can consider the case $k_1 = a_{11} \neq 0$, $k_2 = a_{22} = 0$ and $M = 0$. Then the surface indicatrix has the form (see Fig. 3)

$$Lx'^2 = \pm 1.$$

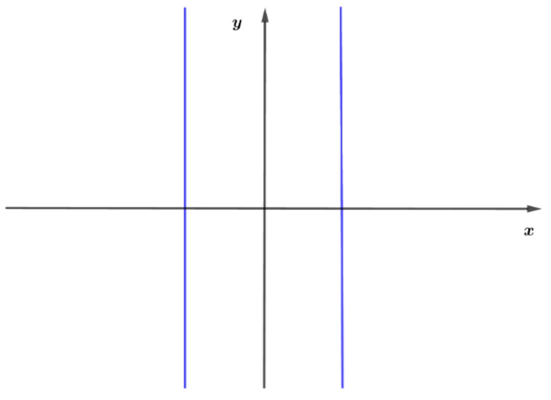

Figure 3.

The above example shows that these two possible kinds of parabolic surface points have different geometrical representations. Moreover, the motion of the tangent plane cannot transform them into each other. Therefore, these cases must be considered separately, i.e., they must be considered as different.

In the first case when $k_1 = a_{11} = 0$, $k_2 = a_{22} \neq 0$, the point will be called a parabolic point of the surface. The second case, when $k_1 = a_{11} \neq 0$, $k_2 = a_{22} = 0$ and $M = 0$, the point will be called a special parabolic point of the surface.

Definition 3. A surface, all points of which are parabolic (especially parabolic), let us call it parabolic (especially parabolic).

For a surface where all points are special parabolic points, our mean curvature $2H = 0$ is equal. In parabolic points, the mean curvature is not $2H \neq 0$.

This leads to the following theorem.

Theorem 1. A special parabolic surface is minimal in Galilean space.

We show the proof of these concepts in the figures. First consider the vector equation of these surfaces and their graph.

$\vec{r_1}(u,v) = u\vec{i} + v\vec{j} + u^2\vec{k}$; $\vec{r_2}(u,v) = u\vec{i} + v\vec{j} + v^2\vec{k}$. $D\{-1 \leq u \leq 1, \ -1 \leq v \leq 1\}$ (1) is the area of definition (see Fig. 4)

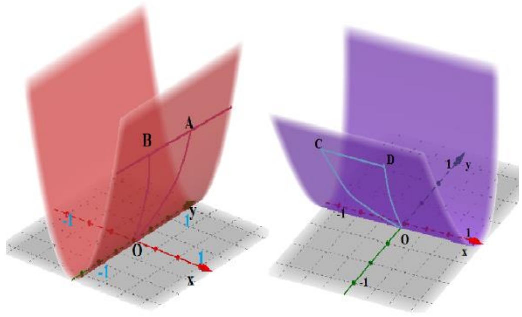

Figure 4.

Now consider the curvature of these surfaces to the plane Oxy, that is, the unambiguous mapping of the surface to the plane with preserving the distances between the corresponding points and the distance order.

The first surface $\vec{r_1}(u,v)$ is mapped to the region D is isometric to Galilean space. The second surface $\vec{r_2}(u,v)$ is mapped to the area $D^*\{-1 \leq u \leq 1, \ -l \leq v \leq l\}$. Here $l = \int_{-1}^{1} \sqrt{1+4v^2}\, dv$ is the length of the parabola.

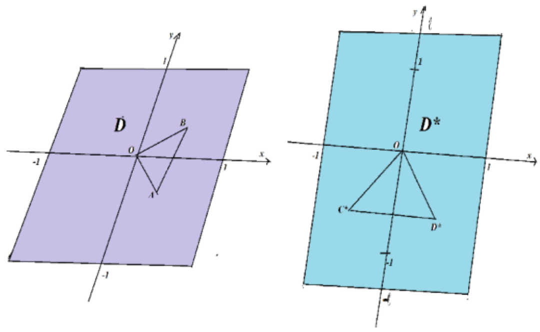

Figure 5.

The peculiarity of the surface isometry of Galilean space can be seen in the mapping of triangles OAB, OCD to the corresponding surfaces. In Galilean space triangles OAB and OA^*B^*, also OCD and OC^*D^* are equal to each other (see Fig. 5)

Therefore, the area of the expansion of the first surface is equal to $S = 4$. The width of the 2nd surface is equal to the area $S = 4l$. Therefore, the surface 1 is smaller than the surface 2. (Surface spreads retain the surface area of the surfaces.)

Assertion 1. Cyclic surfaces are minimal Galilean surfaces.

The mean curvature of the cyclic surface is equal to $2H = 0$. Therefore, the surface is a minimum surface. Let's present the equation of this minimal surface.

It is known [13,14] that for the points of a surface to be cyclic, it is necessary and sufficient that the coefficient of the second fundamental form N turns to zero, i.e.:

$$N = \frac{y_{vv}z_v - z_{vv}y_v}{\sqrt{y_v^2 + z_v^2}} = 0 \ , M = \frac{y_{uv}z_v - z_{uv}y_v}{\sqrt{y_v^2 + z_v^2}} \neq 0.$$

The question naturally arises whether there exist surfaces of Galilean space satisfying these conditions: $N = 0 \ \ M \neq 0$.

To answer this question, let us define the general form of functions $y(u,v)$, $z(u,v)$ satisfying these conditions.

From the above we have

$$y_{vv}z_v - z_{vv}y_v = 0 \text{ and } y_{uv}z_v - z_{uv}y_v \neq 0,$$
$$\text{or } z^2{}_v(\frac{y_v}{z_v})_v = 0 \text{ and } z^2{}_v(\frac{y_v}{z_v})_u \neq 0.$$

These relations show that relation $\frac{y_v}{z_v}$ depends only on parameter u, i.e. $\frac{y_v}{z_v} = F(u)$. We easily obtain the expressions for $y(u,v)$ and $z(u,v)$ in the form

$$y = f_1(u)v + f_3(u)\,,\ z = f_2(u)v + f_4(u).$$

From the obtained results we have the following equation

$$\vec{r} = \vec{r}(u,v) = ui + (f_1(u)v + f_3(u))j + (f_2(u)v + f_4(u))k.$$

This is a general form of the equation of a minimal surface whose points are cyclic.

Let the surface F be given by an explicit equation $z = f(x,y)$.

Theorem 2. If condition $f_{yy} = 0$ is satisfied for surface F in Galilean space G_3, then it is a minimal surface.

Proof: Let's find the coefficients of the first and second fundamental form of the surface from formulas (2), (4):

$$E = 1, G = \sqrt{1 + f_y{}'},$$
$$L = \frac{f_{xx}}{\sqrt{1+f_y^2}}, M = \frac{f_{xy}}{\sqrt{1+f_y^2}}, N = \frac{f_{yy}}{\sqrt{1+f_y^2}}.$$

Now, using (3), we calculate the mean curvature of the surface F

$$2H = \frac{f_{yy}}{(1+f_y^2)^{\frac{3}{2}}}.$$

It follows that this surface is minimal if and only if $f_{yy} = 0$.

Definition 4. A linear surface whose rectilinear formations are parallel to the same plane is called a Catalan surface.

Consider an arbitrary tangent surface:

$$\vec{r}(u,v) = \vec{\alpha}(u) + v\vec{\beta}(u).$$

It is seen that $f_{yy} = 0$ for this surface. This surface will be the minimal surface.

Let the surface be given by the following vector equation:

$$r(u,v) = u\vec{\mathbf{i}} + y(u,v)\vec{\mathbf{j}} + z(u,v)\vec{\mathbf{k}}, \qquad (5)$$

where $y(u,v)$ and $z(u,v)$ are functions with continuous partial derivatives. In order for the surface under consideration (3) to be a minimum surface, the following condition must be met

$$y_{vv}z_v - z_{vv}y_v = 0. \qquad (6)$$

Let's present the equation of the surface (5) in Galilean space, satisfying condition (6). To solve equation (6), we write the expression as follows

$$y_{vv}z_v = z_{vv}y_v.$$

In this equation, the derivatives are proportional

$$\frac{y_{vv}}{y_v} = \frac{z_{vv}}{z_v}.$$

From this proportionality comes the following ratio

$$\frac{d}{dv}(\ln|y_v|) = \frac{d}{dv}(\ln|z_v|).$$

From this

$$\ln|y_v| - \ln|z_v| = C(u),$$

where $C(u)$ is an arbitrary function that depends only on u.This follows from

$$|y_v| = k(u)|z_v|,$$

where $k(u) = e^{C(u)}$ is an arbitrary positive function. According to the above, there is the following relationship between $y(u,v)$ and $z(u,v)$:

$$y(u,v) = f(u) + k(u)z(u,v),$$

where $f(u)$ and $k(u)$ are arbitrary differentiable functions. In general, the vector equation of a surface is

$$r(u,v) = u\vec{\mathbf{i}} + (f(u) + k(u)z(u,v))\vec{\mathbf{j}} + z(u,v)\vec{\mathbf{k}}. \quad (7)$$

If the surface is given by equation (7), then the surface is minimal.

IV. Conclusion

In this article, the minimum surfaces in Galilean space are investigated. The parabolic point of the surface has been studied. The features of geometry in the vicinity of parabolic and special parabolic points of the surface have been revealed. It has been proven that special parabolic and cyclic surfaces have a minimum surface and their equation has been shown. The conditions for the minimum division of surfaces given by the explicit and vector equations in Galilean space have been shown.

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